Unit I: Locating the roots of the equations ①

Chapter I:

1. Bisection method:

This method is based on the repeated application of the intermediate value theorem. The method consists of locating the root of the equation $f(x)=0$ between a and b. $(a<b)$

If $f(x)$ is continuous in $[a,b]$ and $f(a)$ & $f(b)$ are of opposite signs then there is a root between a and b.

Let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root is $x_1 = \frac{a+b}{2}$.

If $f(x_1)=0$, then x_1 is the root of the eqn. $f(x)=0$. Otherwise the root lies between a & x_1 and x_1 & b accordingly as $f(x_1)$ is positive or negative. Repeat the process.

Problems:

1) Find the root of the equation $f(x) = x^3-4x-9=0$ using bisection method in 4 stages.

Ans:

$$f(x) = x^3-4x-9.$$
$$f(0) = 0-4(0)-9 = -9 < 0.$$
$$f(1) = 1^3-4(1)-9 = -12 < 0.$$
$$f(2) = 2^3-4(2)-9 = -9 < 0$$
$$f(3) = 3^3-4(3)-9 = 6 > 0$$

$f(2)$ is $-ve$ and $f(3)$ is $+ve$.

$\therefore$ root lies between 2 and 3.

$x_1 = \frac{a+b}{2}$.

I stage: $x_1 = \frac{2+3}{2} = \frac{5}{2} = 2.5$.

$f(a)f(x_1) = f(2)f(2.5) = (-9)(-3.375) = +ve$

$f(2.5)$
$= (2.5)^3 - 4(2.5) - 9$
$= -3.375$

$f(x_1)f(b) = f(2.5)f(3) = (-3.375)(6) = -ve.$

$\because f(2.5)f(3)$ is $-ve$. Root lies between 2.5 and 3

<u>II stage:</u> $x_2 = \dfrac{\overset{a}{2.5} + \overset{b}{3}}{2} = 2.75$

$f(a)f(x_2) = f(2.5)f(2.75) = (-3.375)(0.7969)$
$= -ve.$

$f(2.75)$
$= (2.75)^3 - 4(2.75) - 9$
$= 0.7968$

$f(x_2)f(b) = f(2.75)f(3) = (0.7969)(6) = +ve.$

Since $f(2.5)f(2.75)$ is $-ve$. Root lies between 2.5 and 2.75

<u>III stage:</u> $x_3 = \dfrac{\overset{a}{2.5} + \overset{b}{2.75}}{2} = 2.625$

$f(a)f(x_3) = f(2.5)f(2.625) = (-3.375)(-1.412)$
$= +ve.$

$f(2.625)$
$= (2.625)^3 - 4(2.625) - 9$
$= -1.412$

$f(x_3)f(b) = f(2.625)f(2.75) = (-1.412)(0.7968)$
$= -ve.$

Since $f(2.625)f(2.75)$ is $-ve$. Root lies between 2.625 and 2.75

$$x_4 = \frac{2.625 + 2.75}{2} = \underline{\underline{2.6875}}.$$

$\therefore$ Root of the equation is 2.6875

2) <u>Bisection method</u> : find the root of the equation $f(x) = x^3 - 2x - 5$ in four stages correct to 3 decimal places. (2)

<u>Ans</u>: $f(x) = x^3 - 2x - 5$

$f(0) = -5 < 0$

$f(1) = -6 < 0$

$f(2) = -1 < 0$ −ve

$f(3) = 16 > 0$ +ve

$\therefore$ root lies between $(2, 3)$

<u>stage 1</u>: $x_1 = \frac{\overset{a}{2} + \overset{b}{3}}{2} = 2.5$

$f(a) = f(2) = 2^3 - 2(2) - 5 = -1 < 0$

$f(x_1) = f(2.5) = 2.5^3 - 2(2.5) - 5 = 5.625 > 0$

$f(b) = f(3) = 3^3 - 2(3) - 5 = 16 > 0$

$f(2) f(2.5) < 0$ $\therefore$ root lies between $(2, 2.5)$

<u>II</u>: $x_2 = \frac{\overset{a}{2} + \overset{b}{2.5}}{2} = 2.25$

$f(a) = f(2) = 2^3 - 2(2) - 5 = -1 < 0$

$f(x_2) = f(2.25) = 2.25^3 - 2(2.25) - 5 = 1.8906 > 0$

$f(b) = f(2.5) = 2.5^3 - 2(2.5) - 5 = 5.625 > 0$

$f(2) f(2.25) < 0$ $\therefore$ lies between $(2, 2.25)$.

<u>III</u>: $x_3 = \frac{2 + 2.25}{2} = 2.125$.

$f(2.125)$
$= (2.125)^3 - 2(2.125) - 5$
$= 0.3457$.

$f(a) f(x_3) = f(2) f(2.25) = (-1)(1.8906)$
$= -ve$.

$f(x_3) f(b) = f(2.125) f(2.25) = (0.3457)(1.8906)$
$= +ve$.

The root lies between 2 and 2.25

$$x_4 = \frac{2+2.25}{2} = \underline{\underline{2.125}}$$

$\therefore$ Root of the equation is 2.125.

2) <u>Newton-Raphson method:</u>

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad n = 0, 1, 2, \dots$$

1) Use N-R method to find a real root of the equation $f(x) = x^3 - 5x + 1$. Perform 4 iteration.

<u>Ans:</u>

$f(x) = x^3 - 5x + 1$

$f(0) = 0 - 5(0) + 1 = 1 > 0$

$f(1) = 1^3 - 5(1) + 1 = -3 < 0$

$f(0)f(1) < 0$. The root lies between 0 and 1.

Let $x_0 = 0.5$.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$$\boxed{\begin{aligned} f(x) &= x^3 - 5x + 1 \\ f'(x) &= 3x^2 - 5 \end{aligned}}$$

<u>I stage</u>: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$= 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$\boxed{\begin{aligned} f(0.5) &= 0.5^3 - 5(0.5) + 1 \\ &= -1.375 \\ f'(0.5) &= 3(0.5)^2 - 5 \\ &= -4.25 \end{aligned}}$$

$$= 0.5 - \frac{(-1.375)}{(-4.25)} = \underline{\underline{0.17647}}$$

<u>II stage</u>: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$$= 0.1765 - \frac{f(0.1765)}{f'(0.1765)}$$

$$= 0.1765 - \frac{0.1229}{(-4.9065)}$$

$$= \underline{0.2015}$$

$f(0.1765)$
$= (0.1765)^3 - 5(0.1765) + 1$
$= 0.1229.$

$f'(0.1765)$
$= 3(0.1765)^2 - 5$
$= -4.9065.$

III stage: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.2015 - \frac{f(0.2015)}{f'(0.2015)}.$$

$$= 0.2015 - \frac{0.0006813}{(-4.8782)}$$

$$= \underline{\underline{0.2016}}.$$

$f(0.2015)$
$= (0.2015)^3 - 5(0.2015) + 1$
$= \underline{0.0006813}$

$f'(0.2015)$
$= 3(0.2015)^2 - 5$
$= -4.8782$

IV stage: $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$= 0.2016 - \frac{f(0.2016)}{f'(0.2016)}$$

$$= 0.2016 - \frac{0.0001935}{(-4.8780)}$$

$$= \underline{\underline{0.2016}}.$$

$f(0.2016)$
$= (0.206)^3 - 5(0.2016) + 1$
$= 0.00019355$

$f'(0.2016)$
$= 3(0.2016)^2 - 5$
$= -4.\underline{8780}$

$\therefore$ The root of the equatis is $\underline{\underline{0.2016}}$.

2) Perform 4 iteration of the N-R method to obtain the approximate value of $(17)^{1/3}$ starting with initial approximation $x_0=2$.

Ans: Let $x=(17)^{1/3}$

$x^3=17 \Rightarrow x^3-17=0$

Let $f(x)=x^3-17$

given $x_0=2$.

$f(x)=x^3-17$

$f'(x)=3x^2$

I stage: $x_1=x_0-\frac{f(x_0)}{f'(x_0)}$

$=2-\frac{f(2)}{f'(2)}=2-\frac{(-9)}{(12)}$

$=2.75$

$f(2)=2^3-17$

$=-9$

$f'(2)=3(2)^2=12$

II stage: $x_2=x_1-\frac{f(x_1)}{f'(x_1)}$

$=2.75-\frac{f(2.75)}{f'(2.75)}$

$=2.75-\frac{3.7968}{22.6875}$

$=2.58264$

$f(2.75)=2.75^3-17$

$=3.7968$

$f'(2.75)=3(2.75)^2$

$=22.6875$

III stage: $x_3=x_2-\frac{f(x_2)}{f'(x_2)}=2.58264-\frac{f(2.58264)}{f'(2.58264)}$

$=2.58264-\frac{0.22644}{20.0100}$

$=2.571323$

IV stage: $x_4=x_3-\frac{f(x_3)}{f'(x_3)}=2.571323-\frac{f(2.571323)}{f'(2.571323)}$

$=2.571323-\frac{0.000835}{7.713969}=2.5712$

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3) Secant method :

$$\boxed{x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] f(x_n)}$$

1) Use secant method to determine the root of the equation $f(x) = x^3 - 2x - 5 = 0$ in $(2,3)$. Perform 3 iterations.

Ans: $x_0 = 2$ and $x_1 = 3$

$f(3) = 3^3 - 2(3) - 5 = 16$

$f(2) = 2^3 - 2(2) - 5 = -1$

Stage 1: $x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1)$

$$= 3 - \left[\frac{3-2}{f(3) - f(2)}\right] f(3)$$

$$= 3 - \left[\frac{1}{16 - (-1)}\right] 16$$

$$= 3 - \left[\frac{1}{17}\right] 16 = 3 - 0.9411 = 2.0588$$

Stage 2: $x_3 = x_2 - \left[\frac{x_2 - x_1}{f(x_2) - f(x_1)}\right] f(x_2)$

$f(2.0588) = 2.0588^3 - 2(2.0588) - 5 = -0.39075$

$$= 2.0588 - \left[\frac{2.0588 - 3}{f(2.0588) - f(3)}\right] f(2.0588)$$

$$= 2.0588 - \left[\frac{-0.9412}{-0.39075 - 16}\right] (-0.39075)$$

$$= 2.0588 - \frac{(-0.9412)}{(-16.39075)} (-0.39075)$$

$$= 2.0588 + (0.0224378) = \underline{\underline{2.08123}}$$

2) Use secant method to find the root of the equation $\cos x - xe^x = 0$ in $(0,1)$.

Ans: Let $x_0 = 0 \quad x_1 = 1$

$f(0) = \cos 0 - 0e^0$
$= 1$

$f(1) = \cos 1 - 1e^1$
$= -2.17797$

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right] f(x_1)$$

$$= 1 - \left[\frac{1-0}{f(1) - f(0)}\right] f(1)$$

$$= 1 - \left[\frac{1}{-2.17797 - 1}\right] (-2.17797)$$

$$= 1 - \left[\frac{1}{-3.17797}\right] (-2.17797)$$

$$= \underline{0.341666}$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.341666 - \left[\frac{0.341666 - 1}{f(0.341666) - f(1)}\right] f(0.341666)$$

$$= 0.341616 - \left[\frac{-0.685333}{0.46137 - (-2.17797)}\right] (0.46137)$$

$$= 0.341666 - \left(\frac{-0.685333}{2.6393445}\right) (0.46137)$$

$$= \underline{0.461465}$$

Chapter 2: Floating point representation and errors

The numbers in the computer word can be stored in two forms.

1) Fixed-point form
2) Floating point form.

In a fixed point form a t digit number is assumed to have its decimal point at the left-hand end of the word. All numbers are assumed to be less than 1.

In the decimal system any real number x can be represented in Normalised floating point form. as $x = \pm 0.d_1 d_2 \dots \times 10^n$.

Machine numbers: The real numbers that are representable in a computers are called machine numbers.

1) subtract 0.5424 E-99 from 0.5452 E-99.

Ans:

$$\begin{array}{r} .5452\ E{-99} \\ .5424\ E{-99} \\ \hline .0028\ E{-99} \end{array}$$

2) Find the sum of 0.123×10^3 and $.456 \times 10^2$

Ans:

$$\begin{array}{r} 0.123 \times 10^3 \\ 0.0456 \\ \hline 0.1686 \times 10^3 \end{array}$$

3) subtract $\cdot 9432E-4$ from $\cdot 5452E-3$

Ans:

$$\begin{array}{r} \cdot 5452E-3 \\ \cdot 09432\ E-3 \\ \hline \cdot 45088 \end{array}$$

4) Multiply $\cdot 5543E12 \times \cdot 4111E-15$

$= \overline{\cdot 2278E-3}$

5) $\cdot 1111E51 \times \cdot 4444E50$

$= \cdot 04937284E\ 101$

$= \overline{\cdot 4937E100}$

Single-precision floating point form:-

A machine number in single-precision floating point form

$(-1)^S \times 2^{(\text{exponent}-\text{bias})} \times (1+\text{signification})$

$(-1)^S \times 2^{(c-127)} \times (1\cdot f)_2 \longrightarrow$ 32 bits

Double-precision floating point form:

$(-1)^S \times 2^{c-1023} \times (1\cdot f)_2 \longrightarrow$ 64 bits

* where the significand provides the significant bits of the number.

* exponent specifies the power of 2. and bias is the value that is used to control the range of the number left most bit used for sign of the mantissa.

S=0 corresponds to + sign

S=1 corresponds to - ve sign.

1) Determine the single-precision machine representation of the decimal no. 52.234375 in both single and double precision

Ans: $(52)_{10} = (110100)_2$

$$
\begin{array}{l}
2\,|\,52 \\
2\,|\,26 - 0 \\
2\,|\,13 - 0 \\
2\,|\,6 - 1 \\
2\,|\,3 - 0 \\
\;\;\;1 - 1
\end{array}
$$

$(.234375)_{10} = (001111)_2$

$(52.234375)_{10} = (110100.001111)_2$

$= (1.\underline{101\ 000\ 011}\ 110)_2 \times 2^5$

$$
\begin{array}{ll}
0.234375 \times 2 & \\
0.46875 \times 2 & 0 \\
0.9375 \times 2 & 0 \\
0.875 \times 2 & 1 \\
0.75 \times 2 & 1 \\
0.5 \times 2 & 1 \\
1 & 0
\end{array}
$$

$2^{c-127} = 2^5$

$c - 127 = 5$

$c = 132$

$= (10000100)_2$

$$
\begin{array}{l}
2\,|\,132 \\
2\,|\,66 - 0 \\
2\,|\,33 - 0 \\
2\,|\,16 - 1 \\
2\,|\,8 - 0 \\
2\,|\,4 - 0 \\
2\,|\,2 - 0 \\
\;\;\;1 - 0
\end{array}
$$

$\therefore (52.234375)_{10}$

$= (\underbrace{0}_{1}\underbrace{100\ 0010\ 0}_{8}\underbrace{101\ 0000\ 1111\ 0000\ 0000\ 0000}_{23})_2$

Double-precision:

$c - 1023 = 5$

$c = 1028$

$= [10000000100]_2$

$$
\begin{array}{l}
2\,|\,1028 \\
2\,|\,514 - 0 \\
2\,|\,257 - 0 \\
2\,|\,128 - 1 \\
2\,|\,64 - 0 \\
2\,|\,32 - 0 \\
2\,|\,16 - 0 \\
\;\;\;8 - 0
\end{array}
$$

2)

~~52~~ 492.~~7~~88125

$(492)_{10} = (111101100)_2$

$(0.788125)_{10} =$

$(11001001)_2$

$$\begin{array}{r|l} 2 & 492 \\ 2 & 246-0 \\ 2 & 123-0 \\ 2 & 61-1 \\ 2 & 30-1 \\ 2 & 15-0 \\ 2 & 7-1 \\ 2 & 3-1 \\ & 1-1 \end{array}$$

0.788125×2

0.57625×2 1

0.1525×2 1

0.305×2 0

0.61×2 0

0.22×2 1

0.44×2 0

0.88×2 0

0.76×2 1

$(492.788125)_{10}$

$= (1111.01100.11001001)_2$

$= (1.1110110011001001)_2 \times 2^8$

$c - 127 = 8$

$c = 127 + 8$

$= 135$

$= (10000111)_2$

$$\begin{array}{r|l} 2 & 135 \\ 2 & 67-1 \\ 2 & 33-1 \\ 2 & 16-1 \\ 2 & 8-0 \\ 2 & 4-0 \\ 2 & 2-0 \\ & 1-0 \end{array}$$

$(0\ 10.00\ 0111\ 1110\ 1100\ 1100\ 1001\ 0000\ 000)_2$

Errors in representing numbers:-

1. Error = True value - approximate value.

2. Relative error $= \dfrac{|\text{Error}|}{|\text{True value}|}$

3. Absolute error = |Error|

4. Inherent error: It is error already present in the statement of the problem before its solution.

5. Round of error : This is a quantity which must be added to the finite representation of a computed number in order to make it the true representation of that number.

Intermediate value theorem: If $f(x)$ is continuous on $[a, b]$ and $f(a)f(b)<0$, then the equation $f(x)=0$ has at least one real root in the interval (a, b).